Factorisation of separable partial differential equations

1987 J. Phys. A: Math. Gen. 20 4577

(http://iopscience.iop.org/0305-4470/20/14/006)

J. Phys. A: Math. Gen. **20** (1987) 4577-4585. Printed in the UK

Factorisation of separable partial differential equations

Mayer Humi

Mathematical Sciences Department, Worcester Polytechnic Institute, Worcester, MA 01609, USA

Received 16 December 1986, in final form 24 February 1987

Abstract. In this paper we study the factorisation of partial differential equations which admit a separation of variables. As a particular application we study the algebraic structure of the resulting raising and lowering operators for the spherical harmonics in three and four dimensions.

1. Introduction

The factorisation method for ordinary differential equations [1, 2] plays a central role in various mathematical [3] and physical [4] applications. Recently, however, this method has been extended to systems of second-order equations [5, 6]. The referee of one of these papers [6] suggested that in view of some recent applications of the factorisation method to supersymmetric models [7, 8] it would be appropriate to consider the extension of these methods to partial differential equations (PDE). Raising and lowering operators for such equations were considered by Boyer and Miller [9] who showed that if these operators are of the first order then the resulting equations are trivial. Nevertheless from a practical point of view many PDE which are important in applications are amenable to separation of variables. It is our purpose in this paper to show how this feature can be used to construct first-order differential operators which raise and lower the indices which characterise the eigenfunctions of such PDE (§ 2).

As a specific application of this technique we construct these operators for the spherical harmonics in three and four dimensions (which are related to the hydrogen atom problem [10, 11]). Furthermore, we study the Lie algebraic structure which is generated by the raising and lowering operators for all the indices which characterise these functions (§ 3).

We would like to note that in order to obtain raising and lowering operators which are independent of the indices themselves one is led naturally to the introduction of 'spurious variables' [3, 12], namely to increase the dimension of the space on which these operators act. Although this spurious dimension can be disposed of in the final step the need for such an extension might have some independent physical implications.

2. Raising and lowering operators for separable PDE

The method we present here to construct raising and lowering operators is applicable to separable PDE with any number of independent variables. However, for the clarity

0305-4470/87/144577+09\$02.50

of the presentation we shall illustrate it only through a prototype example in two variables.

Consider the equation

$$\left[\frac{\partial^2}{\partial y^2}+r_2(y)\left(\frac{\partial^2}{\partial x^2}+r_1(x,m)\right)+\lambda\right]\psi=0. \tag{2.1}$$

Using separation of variables, i.e. introducing $\psi=\varphi^{(1)}(x)\varphi^{(2)}(y)$ we obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+r_1(x,m)+f(\mu)\right)\varphi^{(1)}=0 \tag{2.2}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-r_2(y)f(\mu)+\lambda\right)\varphi^{(2)}=0 \tag{2.3}$$

where $f(\mu)$ is a separation constant. Assuming no hidden degeneracy we infer then that the eigenfunctions of (2.1) are labelled by three indices:

$$\psi=\psi(\lambda,\mu,m)$$

(we drop the dependence on x, y). If (2.2) and (2.3) are factorisable [1] then one can construct from (2.2) raising and lowering operators for ψ in m (type I operators in the nomenclature of [1]), i.e. we can find functions $k_1(x,m)$ and $L_1(m)$ so that

$$\left(k_1(x,m+1)-\frac{\partial}{\partial x}\right)\psi(\lambda,\mu,m)=(f(\mu)-L_1(m+1))^{1/2}\psi(\lambda,\mu,m+1) \tag{2.4}$$

$$\left(k_1(x,m)+\frac{\partial}{\partial x}\right)\psi(\lambda,\mu,m)=(f(\mu)-L_1(m))^{1/2}\psi(\lambda,\mu,m-1). \tag{2.5}$$

Similarly we can obtain from (2.3) type II operators to raise and lower λ, i.e. we can find $k_2(y,\lambda)$ and $L_2(\lambda)$ so that

$$\left(k_2(y,\lambda+1)-\frac{\partial}{\partial y}\right)\psi(\lambda,\mu,m)=(g_2(\mu)-L_2(\lambda+1))^{1/2}\psi(\lambda+1,\mu,m) \tag{2.6}$$

$$\left(k_2(y,\lambda)+\frac{\partial}{\partial y}\right)\psi(\lambda,\mu,m)=(g_2(\mu)-L_2(\lambda))^{1/2}\psi(\lambda-1,\mu,m) \tag{2.7}$$

where $g_2(\mu)$ is a function which is determined through the factorisation process.

To find raising and lowering operators for μ it would have been natural to find type II operators for (2.2) and type I operators for (2.3) and then multiply them out to obtain the desired result on ψ. Thus we can find $k_3(x,\mu)$, $k_4(y,\mu)$ and $L_3(\mu)$, $L_4(\mu)$ so that

$$H_1\varphi^{(1)}(\mu,m)=\left(k_3(x,\mu+1)-\frac{\partial}{\partial x}\right)\varphi^{(1)}(\mu,m)=\alpha(m,\mu+1)\varphi^{(1)}(\mu+1,m) \tag{2.8}$$

$$H_2\varphi^{(1)}(\mu,m)=\left(k_3(x,\mu)+\frac{\partial}{\partial x}\right)\varphi^{(1)}(\mu,m)=\alpha(m,\mu)\varphi^{(1)}(\mu-1,m) \tag{2.9}$$

$$H_3\varphi^{(2)}(\lambda,\mu)=\left(k_4(y,\mu+1)-\frac{\partial}{\partial y}\right)\varphi^{(2)}(\lambda,\mu)=\beta(\lambda,\mu+1)\varphi^{(2)}(\lambda,\mu+1) \tag{2.10}$$

$$H_4\varphi^{(2)}(\lambda,\mu)=\left(k_4(y,\mu)+\frac{\partial}{\partial y}\right)\varphi^{(2)}(\lambda,\mu)=\beta(\lambda,\mu)\varphi^{(2)}(\lambda,\mu-1) \tag{2.11}$$

where

$$\alpha(m, \mu) = (g_1(m) - L_3(\mu))^{1/2}$$
$$\beta(\lambda, \mu) = (\lambda - L_4(\mu))^{1/2}.$$

It is obvious then that

$$H_1 H_3 \psi(\lambda, \mu, m) \sim \psi(\lambda, \mu+1, m) \tag{2.12}$$
$$H_2 H_4 \psi(\lambda, \mu, m) \sim \psi(\lambda, \mu-1, m). \tag{2.13}$$

However, these are second-order operators and therefore not derivations. (Another approach will be to construct raising and lowering operators with no differential realisation by taking the formal square root of these or some related operators as was done recently [11] for the non-relativistic Coulomb problem.) To overcome the difficulties pointed out above we now observe that from (2.8)-(2.11) it follows that

$$(k_3(x, \mu) + k_3(x, \mu+1))\varphi^{(1)}(\mu, m) = \alpha(m, \mu)\varphi^{(1)}(\mu-1, m) + \alpha(m, \mu+1)\varphi^{(1)}(\mu+1, m)$$
$$(k_4(y, \mu) + k_4(y, \mu+1))\varphi^{(2)}(\lambda, \mu) = \beta(\lambda, \mu)\varphi^{(2)}(\lambda, \mu-1) + \beta(\lambda, \mu+1)\varphi^{(2)}(\lambda, \mu+1)$$
$$\left(k_3(x, \mu+1) - k_3(x, \mu) - 2\frac{\partial}{\partial x}\right)\varphi^{(1)}(\lambda, \mu)$$
$$= \alpha(m, \mu+1)\varphi^{(1)}(\mu+1, m) - \alpha(m, \mu)\varphi^{(1)}(\mu-1, m)$$
$$\left(k_4(y, m+1) - k_4(y, \mu) - 2\frac{\partial}{\partial y}\right)\varphi^{(2)}(\lambda, \mu)$$
$$= \beta(\lambda, \mu+1)\varphi^{(2)}(\lambda, \mu+1) - \beta(\lambda, \mu)\varphi^{(2)}(\lambda, \mu-1).$$

Hence we infer that

$$\begin{aligned} P\psi(\lambda, \mu, m) = \Bigg[&(k_3(x, \mu) + k_3(x, \mu+1))\left(k_4(y, \mu+1) - k_4(y, \mu) - 2\frac{\partial}{\partial y}\right) \\ &+ (k_4(y, \mu) + k_4(y, \mu+1))\left(k_3(x, \mu+1) - k_3(x, \mu) - 2\frac{\partial}{\partial x}\right)\Bigg]\psi(\lambda, \mu, m) \\ &= 2\alpha(m, \mu+1)\beta(\lambda, \mu+1)\psi(\lambda, \mu+1, m) - 2\alpha(m, \mu)\beta(\lambda, \mu)\psi(\lambda, \mu-1, m). \end{aligned} \tag{2.14}$$

Thus the operator P is a first-order differential operator which raises and lowers μ at the same time. However, one can commute this operator with those obtained earlier to raise and lower m and λ in order to generate new operators which also raise and lower the indices of the eigenfunctions $\psi(\lambda, \mu, m)$. Once this set of operators closes under commutation, i.e. generates a Lie algebra, one obtains the complete set of raising and lowering operators which the eigenvalue problem admits. (However, note that we do not obtain pure raising or lowering operators for μ in this method.)

A cautionary remark is appropriate at this juncture. In the prototype example we studied above we assumed that the separated equations (2.2) and (2.3) are already in a standard form which lends itself to direct application of the factorisation method. In actual practice, however, one usually has to perform certain transformations on the separated equations in order to bring them into standard form and find their corresponding raising and lowering operators. Nevertheless, once one finds these operators for the standard equations one can reduce their action to the solutions of the original equations. Furthermore, one might be careful to use consistent normalisation of these solutions if both factorisations of type I and II are being considered for the same equation. Although these technical details might complicate (or simplify) the expressions for the raising and lowering operators and their action on the eigenfunctions the construction above will remain valid in general.

In the next section we demonstrate this construction for the spherical harmonics in four dimensions. However, in order to introduce the method of spurious coordinates we consider first the spherical harmonics in three dimensions and investigate the algebraic structure that arises when the full set of raising and lowering operators in l, m is considered.

3. Algebraic aspects of factorisable PDE

3.1. Spherical harmonics in three dimensions

The spherical harmonics in three dimensions are the solutions of

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}+l(l+1)\right]Y_{l,m}(\theta,\varphi)=0 \tag{3.1}$$

where $Y_{l,m}(\theta,\varphi)$ can be written as

$$Y_{l,m}(\theta,\varphi)=P_{l,m}(\theta)\,\mathrm{e}^{\mathrm{i}m\varphi}. \tag{3.2}$$

The raising and lowering operators in m are given by

$$J_+=\mathrm{e}^{\mathrm{i}\varphi}\left(-\mathrm{i}\cot\theta\frac{\partial}{\partial\varphi}-\frac{\partial}{\partial\theta}\right) \tag{3.3}$$

$$J_-=\mathrm{e}^{-\mathrm{i}\varphi}\left(\mathrm{i}\cot\theta\frac{\partial}{\partial\varphi}+\frac{\partial}{\partial\theta}\right) \tag{3.4}$$

whose commutator is

$$[J_+,J_-]=2J_3 \tag{3.5}$$

where

$$J_3=-\mathrm{i}\frac{\partial}{\partial\varphi}. \tag{3.6}$$

The action of these operators on the spherical harmonics is given by

$$J_+Y_{l,m}=[(l+m+1)(l-m)]^{1/2}Y_{l,m+1} \tag{3.7}$$

$$J_-Y_{l,m}=[(l+m)(l-m+1)]^{1/2}Y_{l,m-1} \tag{3.8}$$

$$J_3Y_{l,m}=mY_{l,m}. \tag{3.9}$$

We can obtain raising and lowering operators in l by performing a type II factorisation on (3.1) which, after taking into account the proper normalisation of these functions, yields [1]

$$\left((l+1)\cos\theta+\sin\theta\frac{\partial}{\partial\theta}\right)Y_{l,m}=\left(\frac{(l+1-m)(l+1+m)(2l+1)}{(2l+3)}\right)^{1/2}Y_{l+1,m} \tag{3.10}$$

$$\left(l\cos\theta-\sin\theta\frac{\partial}{\partial\theta}\right)Y_{l,m}=\left(\frac{(l-m)(l+m)(2l+1)}{(2l-1)}\right)^{1/2}Y_{l-1,m}. \tag{3.11}$$

These raising and lowering operators in l depend explicitly on l. Therefore, formally they are defined only on a proper subspace generated by $\{Y_{l,m}\}$ for fixed m. To obtain operators that act on the whole space of these functions we introduce a spurious coordinate χ and define

$$\tilde{Y}(\theta, \varphi, \chi) = e^{il\chi} Y_{l,m}(\theta, \varphi). \tag{3.12}$$

The raising and lowering operators in l on these functions then take the form

$$K_+ = e^{i\chi}\left[\left(-i\frac{\partial}{\partial\chi}+1\right)\cos\theta + \sin\theta\frac{\partial}{\partial\theta}\right] \tag{3.13}$$

$$K_- = -e^{-i\chi}\left(i\cos\theta\frac{\partial}{\partial\chi}+\sin\theta\frac{\partial}{\partial\theta}\right) \tag{3.14}$$

whose commutator is

$$[K_+, K_-] = -2K_3 \tag{3.15}$$

where

$$K_3 = -i\,\partial/\partial\chi + \tfrac{1}{2}. \tag{3.16}$$

The operators J_+, J_-, J_3 generate an O(3) algebra in view of (3.5) and

$$[J_3, J_+] = J_+ \qquad [J_3, J_-] = -J_-. \tag{3.17}$$

However, since

$$[K_3, K_+] = K_+ \qquad [K_3, K_-] = -K_- \tag{3.18}$$

we infer from (3.15) and (3.18) that K_+, K_-, K_3 form an O(2, 1) algebra.

To see the algebraic structures that these six operators generate together we form their commutators and add any new operator which is obtained in this way.

Thus, we obtain

$$R_+ = [K_+, J_+] = \exp[i(\varphi+\chi)]\left[\left(i\frac{\partial}{\partial\chi}-1\right)\sin\theta + \cos\theta\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right] \tag{3.19}$$

$$L_- = [K_-, J_-] = -\exp[-i(\varphi+\chi)]\left(i\sin\theta\frac{\partial}{\partial\chi}-\cos\theta\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \tag{3.20}$$

$$L_+ = [K_+, J_-] = -\exp[i(\chi-\varphi)]\left[\left(i\frac{\partial}{\partial\chi}-1\right)\sin\theta + \cos\theta\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right] \tag{3.21}$$

$$R_- = [K_-, J_+] = \exp[i(\varphi-\chi)]\left(i\sin\theta\frac{\partial}{\partial\chi}-\cos\theta\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right). \tag{3.22}$$

The action of these operators on $\tilde{Y}_{l,m}$ is given by

$$R_+\tilde{Y}_{l,m} = -\left(\frac{(l+m+2)(l+m+1)(2l+1)}{(2l+3)}\right)^{1/2}\tilde{Y}_{l+1,m+1} \tag{3.23}$$

$$L_-\tilde{Y}_{l,m} = \left(\frac{(l+m)(l+m-1)(2l+1)}{(2l-1)}\right)^{1/2}\tilde{Y}_{l-1,m-1} \tag{3.24}$$

$$L_+\tilde{Y}_{l,m} = -\left(\frac{(l-m+1)(l-m+2)(2l+1)}{(2l+3)}\right)^{1/2}\tilde{Y}_{l+1,m-1} \tag{3.25}$$

$$R_-\tilde{Y}_{l,m} = \left(\frac{(l-m-1)(l-m)(2l+1)}{(2l-1)}\right)^{1/2}\tilde{Y}_{l-1,m+1}. \tag{3.26}$$

The ten operators $J_\pm$, J_3, $K_\pm$, K_3, $R_\pm$, $L_\pm$ generate the Lie algebra of O(3, 2) [11]. Their non-zero commutation relations (except those given already) are

$$\begin{aligned}
&[R_+, J_-]=2K_+ \qquad [R_+, L_-]=4K_3+4J_3 \qquad [R_+, J_3]=R_+\\
&[R_+, K_-]=2J_+ \qquad [R_+, K_3]=R_+ \qquad [R_-, J_-]=2K_-\\
&[R_-, L_+]=4J_3+4K_3 \qquad [R_-, J_3]=-R_- \qquad [R_-, K_3]=R_-\\
&[R_-, K_+]=2J_+ \qquad [L_+, J_+]=2K_+ \qquad [L_+, J_3]=L_+\\
&[L_+, K_-]=2J_- \qquad [L_+, K_3]=-L_+ \qquad [L_-, J_+]=2K_-\\
&[L_-, J_3]=L_- \qquad [L_-, K_+]=2J_- \qquad [L_-, K_3]=L_-.
\end{aligned} \tag{3.27}$$

3.2. *Spherical harmonics in four dimensions*

These functions are the eigenfunctions of the Laplace operator on the unit ball in four dimensions. In polar coordinates this equation takes the form

$$\left[\frac{\partial^2}{\partial\alpha^2}+2\cot\alpha\frac{\partial}{\partial\alpha}+\frac{1}{\sin^2\alpha}\left(\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)+(n^2-1)\right]\psi=0$$

$$n=1,2,\ldots \tag{3.28}$$

where n is an integer. If we perform a separation of variables

$$\psi_{n,l,m}=N_{n,l}(\alpha)Y_{l,m}(\theta,\varphi) \qquad 0\leqslant l\leqslant n-1,\ -l\leqslant m\leqslant l \tag{3.29}$$

we obtain for $N_{n,l}$ the equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}+2\cot\alpha\frac{\mathrm{d}}{\mathrm{d}\alpha}-\frac{l(l+1)}{\sin^2\alpha}+(n^2-1)\right)N_{n,l}=0 \tag{3.30}$$

while $Y_{l,m}$ satisfy equation (3.1).

The raising and lowering operators in m and l for $Y_{l,m}$ are given as before by (3.3), (3.4), (3.10) and (3.11). The raising and lowering operators in l for $N_{n,l}$ are

$$\left(l\cot\alpha-\frac{\partial}{\partial\alpha}\right)N_{n,l}=[n^2-(l+1)^2]N_{n,l+1} \tag{3.31}$$

$$\left((l+1)\cot\alpha+\frac{\partial}{\partial\alpha}\right)N_{n,l}=(n^2-l^2)N_{n,l-1}. \tag{3.32}$$

To obtain the equivalent of the operator P in § 2, we multiply (3.11) by $\cot\alpha N_{n,l}$, (3.31) by $\cos\theta Y_{l,m}$ and subtract

$$\begin{aligned}
&\left(\cos\theta\frac{\partial}{\partial\alpha}-\sin\theta\cot\alpha\frac{\partial}{\partial\theta}\right)\psi_{n,l,m}\\
&\quad=\left(\frac{(l-m)(l+m)(2l+1)}{(2l-1)}\right)^{1/2}Y_{l-1,m}N_{n,l}\cot\alpha\\
&\quad\quad-[n^2-(l+1)^2]N_{n,l+1}Y_{l,m}\cos\theta.
\end{aligned} \tag{3.33}$$

Using (3.10), (3.11), (3.31) and (3.32) we infer, however, that

$$\cos\theta Y_{l,m}=\left(\frac{(l+1-m)(l+1+m)}{(2l+1)(2l+3)}\right)^{1/2}Y_{l+1,m}+\left(\frac{(l-m)(l+m)}{(2l-1)(2l+1)}\right)^{1/2}Y_{l-1,m} \tag{3.34}$$

$$\cot\alpha N_{n,l}=\left(\frac{n^2-l^2}{(2l+1)^2}\right)^{1/2}N_{n,l-1}+\left(\frac{n^2-(l+1)^2}{(2l+1)^2}\right)^{1/2}N_{n,l+1}. \tag{3.35}$$

Substituting (3.34) and (3.35) in (3.33) then yields

$$P_6\psi_{n,l,m} = \left(\cos\theta\frac{\partial}{\partial\alpha} - \sin\theta\cot\alpha\frac{\partial}{\partial\theta}\right)\psi_{n,l,m}$$

$$= \left(\frac{(l-m)(l+m)(n^2-l^2)}{(2l+1)(2l-1)}\right)^{1/2}\psi_{n,l-1,m}$$

$$-\left(\frac{(l+1-m)(l+1+m)[n^2-(l+1)^2]}{(2l+1)(2l+3)}\right)^{1/2}\psi_{n,l+1,m}. \tag{3.36}$$

Commuting P_6 with J_+, J_- we obtain the following two operators:

$$P_4 = [P_6, J_+] = -\mathrm{e}^{\mathrm{i}\varphi}\left(\sin\theta\frac{\partial}{\partial\alpha} + \cos\theta\cot\alpha\frac{\partial}{\partial\theta} + \frac{\mathrm{i}\cot\alpha}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \tag{3.37}$$

$$P_5 = [P_6, J] = \mathrm{e}^{-\mathrm{i}\varphi}\left(\sin\theta\frac{\partial}{\partial\alpha} + \cos\theta\cot\alpha\frac{\partial}{\partial\theta} - \frac{\mathrm{i}\cot\alpha}{\sin\theta}\frac{\partial}{\partial\varphi}\right). \tag{3.38}$$

(The differential expressions for the operators P_4, P_5, P_6 were first found by Kaufman [13].) The action of these operators on the functions $\psi_{n,l,m}$ is given by

$$P_4\psi_{n,l,m} = \left(\frac{(l+m+1)(l+m+2)(n-l-1)(n+l+1)}{(2l+1)(2l+3)}\right)^{1/2}\psi_{n,l+1,m+1}$$

$$+\left(\frac{(l-m-1)(l-m)(n-l)(n+l)}{(2l-1)(2l+1)}\right)^{1/2}\psi_{n,l-1,m+1} \tag{3.39}$$

$$P_5\psi_{n,l,m} = \left(\frac{(l-m+1)(l-m+2)(n-l-1)(n+l+1)}{(2l+1)(2l+3)}\right)^{1/2}\psi_{n,l+1,m-1}$$

$$+\left(\frac{(l+m-1)(l+m)(n-l)(n+l)}{(2l-1)(2l+1)}\right)^{1/2}\psi_{n,l-1,m-1}. \tag{3.40}$$

The six operators $J_\pm$, J_3 and P_4, P_5, P_6 satisfy the following non-zero commutation relations (in addition to those already calculated):

$$\begin{aligned} &[J_+, P_5] = -2P_6 \qquad &[J_-, P_4] = -2P_6 \\ &[J_3, P_4] = P_4 \qquad &[J_3, P_5] = -P_5. \end{aligned} \tag{3.41}$$

Hence these operators generate, as expected, an O(4) algebra (since (3.28) is the Casimir operator of this group). To obtain raising and lowering operators in all the indices of the eigenfunctions $\psi_{n,l,m}$ we must add to these six operators the raising and lowering operators in n which are (using the correct normalisation for $\psi_{n,l,m}$)

$$\left((n+1)\cos\alpha + \sin\alpha\frac{\partial}{\partial\alpha}\right)\psi_{n,l,m} = \left(\frac{n(n-l)(n+l+1)}{(n+1)}\right)^{1/2}\psi_{n+1,l,m} \tag{3.42}$$

$$\left((n-1)\cos\alpha - \sin\alpha\frac{\partial}{\partial\alpha}\right)\psi_{n,l,m} = \left(\frac{n(n-l-1)(n+l)}{(n-l)}\right)^{1/2}\psi_{n-l,l,m}. \tag{3.43}$$

To obtain operators that act on the whole space of these eigenfunctions we introduce as before a spurious coordinate χ and define

$$\tilde{\psi}(\alpha, \theta, \varphi, x) = \mathrm{e}^{\mathrm{i}nx}\psi_{n,l,m}(\alpha, \theta, \varphi). \tag{3.44}$$

The raising and lowering operators on n take the form

$$P_7 = e^{i\chi}\left[\left(-i\frac{\partial}{\partial\chi}+1\right)\cos\alpha + \sin\alpha\frac{\partial}{\partial\alpha}\right] \tag{3.45}$$

$$P_8 = e^{i\chi}\left[\left(-i\frac{\partial}{\partial\chi}-1\right)\cos\alpha - \sin\alpha\frac{\partial}{\partial\alpha}\right] \tag{3.46}$$

whose commutator is

$$[P_7, P_8] = -2P_9 \tag{3.47}$$

where

$$P_9 = -i\frac{\partial}{\partial\chi}. \tag{3.48}$$

To close the algebra generated by $J_\pm$, J_3 and $P_3, \ldots, P_9$ we now form their commutators and add any new operator which is so obtained. In this fashion we find the operators $P_{10}, \ldots, P_{15}$ whose differential expressions and their action on $\tilde{\psi}_{n,l,m}$ are given in the appendix. Together these fifteen operators $J_\pm$, J_3, $P_4, \ldots, P_{15}$ generate an algebra of O(4, 2).

Remark. Note that after one derives these new operators it is possible to discard the spurious coordinate χ if so desired.

Appendix

In this appendix we give the explicit form of the operators $P_{10} \ldots P_{15}$ and their action on $\tilde{\psi}(n, l, m)$.

$$\begin{aligned}P_{10}\tilde{\psi}(n,l,m) &= [P_7, P_4]\tilde{\psi}(n,l,m)\\ &= -\left(\frac{(l+m+1)(l+m+2)n(n+l+1)(n+l+2)}{(2l+1)(2l+3)(n+1)}\right)^{1/2}\tilde{\psi}(n+1, l+1, m+1)\\ &\quad -\left(\frac{(l-m-1)(l-m)n(n-l)(n-l+1)}{(2l-1)(2l+1)(n+1)}\right)^{1/2}\tilde{\psi}(n+1, l-1, m+1)\end{aligned}$$

$$\begin{aligned}P_{11}\tilde{\psi}(n,l,m) &= [P_7, P_5]\tilde{\psi}(n,l,m)\\ &= -\left(\frac{(l-m+1)(l-m+2)n(n+l+1)(n+l+2)}{(2l+1)(2l+3)(n+1)}\right)^{1/2}\tilde{\psi}(n+1, l+1, m-1)\\ &\quad -\left(\frac{(l+m+1)(l+m)n(n-l+1)(n-l)}{(2l-1)(2l+1)(n+1)}\right)^{1/2}\tilde{\psi}(n+1, l-1, m-1)\end{aligned}$$

$$\begin{aligned}P_{12}\tilde{\psi}(n,l,m) &= [P_7, P_6]\tilde{\psi}(n,l,m)\\ &= \left(\frac{(l-m+1)(l+m+1)n(n+l+1)(n+l+2)}{(2l+1)(2l+3)(n+1)}\right)^{1/2}\tilde{\psi}(n+1, l+1, m)\\ &\quad -\left(\frac{(l-m)(l+m)n(n-l)(n-l+1)}{(2l-1)(2l+1)(n+1)}\right)^{1/2}\tilde{\psi}(n+1, l-1, m)\end{aligned}$$

$$P_{13}\tilde{\psi}(n, l, m) = [P_8, P_4]\tilde{\psi}(n, l, m)$$
$$= \left(\frac{(l+m+1)(l+m+2)n(n-l-1)(n-l-2)}{(2l+1)(2l+3)(n-1)}\right)^{1/2} \tilde{\psi}(n-1, l+1, m+1)$$
$$+ \left(\frac{(l-m-1)(l-m)n(n+1)(n+l-1)}{(2l-1)(2l+1)(n-1)}\right)^{1/2} \tilde{\psi}(n-1, l-1, m+1)$$

$$P_{14}\tilde{\psi}(n, l, m) = [P_8, P_5]\tilde{\psi}(n, l, m)$$
$$= \left(\frac{(l-m+1)(l-m+2)n(n-l-1)(n-l-2)}{(2l+1)(2l+3)(n-1)}\right)^{1/2} \tilde{\psi}(n-1, l+1, m-1)$$
$$+ \left(\frac{(l+m-1)(l+m)n(n+l)(n+l-1)}{(2l-1)(2l+1)(n-l)}\right)^{1/2} \tilde{\psi}(n-1, l-1, m-1)$$

$$P_{15}\tilde{\psi}(n, l, m) = [P_8, P_6]\tilde{\psi}(n, l, m)$$
$$= \left(\frac{(l-m)(l+m)n(n+l)(n+l-1)}{(2l-1)(2l+1)(n-1)}\right)^{1/2} \tilde{\psi}(n-1, l-1, m)$$
$$- \left(\frac{(l-m+1)(l+m+1)n(n-l-1)(n-l-2)}{(2l+1)(2l+3)(n-1)}\right)^{1/2} \tilde{\psi}(n-1, l+1, m).$$

The differential expressions for these operators are

$$P_{10} = \exp[\mathrm{i}(\chi+\varphi)]\left[\sin\theta\sin\alpha\left(\mathrm{i}\frac{\partial}{\partial\chi}-1\right)+\sin\theta\cos\alpha\frac{\partial}{\partial\alpha}+\frac{\cos\theta}{\sin\alpha}\frac{\partial}{\partial\theta}+\frac{\mathrm{i}}{\sin\theta\sin\alpha}\frac{\theta}{\partial\varphi}\right]$$

$$P_{11} = -\exp[\mathrm{i}(\chi-\varphi)]\left[\sin\theta\sin\alpha\left(\mathrm{i}\frac{\partial}{\partial\chi}-1\right)+\sin\theta\cos\alpha\frac{\partial}{\partial\alpha}+\frac{\cos\theta}{\sin\alpha}-\frac{\mathrm{i}}{\sin\theta\sin\alpha}\frac{\partial}{\partial\varphi}\right]$$

$$P_{12} = -\exp(\mathrm{i}\chi)\left[\cos\theta\sin\alpha\left(\mathrm{i}\frac{\partial}{\partial\chi}-1\right)+\cos\theta\cos\alpha\frac{\partial}{\partial\alpha}-\frac{\sin\theta}{\sin\alpha}\frac{\partial}{\partial\theta}\right]$$

$$P_{13} = \exp[\mathrm{i}(\varphi-\chi)]\left[\sin\theta\sin\alpha\left(\mathrm{i}\frac{\partial}{\partial\chi}+1\right)-\sin\theta\cos\alpha\frac{\partial}{\partial\alpha}-\frac{\cos\theta}{\sin\alpha}\frac{\partial}{\partial\theta}-\frac{\mathrm{i}}{\sin\theta\sin\alpha}\frac{\partial}{\partial\varphi}\right]$$

$$P_{14} = \exp[-\mathrm{i}(\varphi+\chi)]\left[\sin\theta\sin\alpha\left(\mathrm{i}\frac{\partial}{\partial\chi}+1\right)-\sin\theta\cos\alpha\frac{\partial}{\partial\alpha}-\frac{\cos\theta}{\sin\alpha}\frac{\partial}{\partial\theta}+\frac{\mathrm{i}}{\sin\theta\sin\alpha}\frac{\partial}{\partial\varphi}\right]$$

$$P_{15} = \exp(-\mathrm{i}\chi)\left[\cos\theta\sin\alpha\left(\mathrm{i}\frac{\partial}{\partial\chi}+1\right)-\cos\theta\cos\alpha\frac{\partial}{\partial\alpha}+\frac{\sin\theta}{\sin\alpha}\frac{\partial}{\partial\theta}\right].$$

References

[1] Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21
[2] Miller W Jr 1968 *Lie Theory and Special Functions* (New York: Academic)
[3] Kaufman B 1966 *J. Math. Phys.* **7** 447
[4] Coulson C A and Joseph A 1967 *Rev. Mod. Phys.* **39** 829, 838
[5] Humi M 1986 *J. Math. Phys.* **27** 76
[6] Humi M 1987 *J. Phys. A: Math. Gen.* **20** 1323
[7] Stedman G E 1986 *Eur. J. Phys.* **6** 225
[8] Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** 2917
[9] Boyer C and Miller W Jr 1974 *J. Math. Phys.* **15** 1484
[10] Bander M and Itzykson C 1986 *Rev. Mod. Phys.* **38** 330
[11] DeLange D L and Raab R E 1986 *Phys. Rev.* A **34** 1650
[12] Weisner L 1955 *Pac. J. Math.* **5** 1033
[13] Kaufman B 1968 private communication